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LETTER TO THE EDITOR

Pseudo-Hermitian versus Hermitian position-dependent-mass Hamiltonians in a perturbative framework

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Abstract
We formulate a systematic algorithm for constructing a whole class of Hermitian position-dependent-mass Hamiltonians which, to lowest order of perturbation theory, allow a description in terms of $\mathcal{PT}$-symmetric Hamiltonians. The method is applied to the Hermitian analogue of the $\mathcal{PT}$-symmetric cubic anharmonic oscillator. A new example is provided by a Hamiltonian (approximately) equivalent to a $\mathcal{PT}$-symmetric extension of the one-parameter trigonometric Pöschl–Teller potential.

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Pseudo-Hermitian Hamiltonians and their subclass of $\mathcal{PT}$-symmetric ones have aroused a great deal of interest since it was observed that some of them may have a real, positive spectrum [1]. Pseudo-Hermiticity of H with respect to a positive-definite (Hermitian and invertible) operator η_+, i.e.,

$$H^\dagger = \eta_+ H \eta_+^{-1}, \tag{1}$$

has been identified as one of the necessary and sufficient conditions for this situation to occur [2]. Any Hamiltonian endowed with such a property is then equivalent to a Hermitian one,

$$h = \rho H \rho^{-1}, \tag{2}$$

where the similarity transformation is implemented by $\rho = \sqrt{\eta_+}$. Further, to any observable o and to any wavefunction $\psi(x) = \langle x|\psi\rangle$ in the Hermitian theory described by h, one can associate an operator $O = \rho^{-1} o \rho$ and a wavefunction $\Psi(x) = \langle x|\rho\psi\rangle$ in the (physical) pseudo-Hermitian theory, respectively.

Recently Jones [3] and, independently, Mostafazadeh [4] constructed the Hermitian analogue h, as well as the pseudo-Hermitian position and momentum operators

$X = \rho^{-1}x\rho$, $P = \rho^{-1}p\rho$, for the $\mathcal{PT}$-symmetric cubic anharmonic oscillator $H = \frac{1}{2}(p^2 + x^2) + \mathrm{i}\epsilon x^3$ (with $\epsilon \in \mathbb{R}$). The latter, which has been shown both numerically [1] and mathematically [5] to have a real, positive and discrete spectrum, can only be treated in perturbation theory [6]. A very interesting outcome of [3, 4] is that to lowest order such a system describes an ordinary quartic anharmonic oscillator with real and positive coupling constants but a position-dependent mass (PDM). As revealed by a more recent study of Bender *et al* [7], this Hermitian PDM theory is however difficult to work out because it leads to divergent Feynman graphs, which must be regulated to obtain the correct answer, whereas the corresponding non-Hermitian $\mathcal{PT}$-symmetric theory is completely free from such difficulties.

At this stage, it is worth mentioning that Hermitian PDM Hamiltonians are attracting a lot of attention due to their relevance in describing the physics of many microstructures of current interest, such as compositionally graded crystals (see [8] and references quoted therein). Several classes of physically-interesting solvable non-Hermitian potentials have also been generated [9–11] in a PDM background by employing various techniques, such as the point canonical transformations or Lie algebraic methods, or using ideas from supersymmetric quantum mechanics. In particular, constructions of $\mathcal{PT}$-symmetric potentials have been carried out for different choices of mass functions. These include the $\mathcal{PT}$-symmetric Scarf potential [9] and the $\mathcal{PT}$-symmetric oscillator model [10]. Even the PDM version of the complex Morse potential [12], which is known to be pseudo-Hermitian [13], has been obtained [10].

In view of all these considerations, it may prove interesting to see under which conditions a Hermitian PDM Hamiltonian may be approximately equivalent to a non-Hermitian $\mathcal{PT}$-symmetric one, which, according to the experience gained in [7], would presumably be easier to handle. In the spirit of [3, 4], this is tantamount to determining those $\mathcal{PT}$-symmetric Hamiltonians,

$$H = H_0 + \varepsilon H_1 \qquad H_0 = \frac{p^2}{2m_0} + V^{(\mathrm{r})}(x) \qquad H_1 = \mathrm{i}V^{(\mathrm{i})}(x) \tag{3}$$

with $\varepsilon \in \mathbb{R}$, $V^{(\mathrm{r})}(x) = V^{(\mathrm{r})}(-x) \in \mathbb{R}$, $V^{(\mathrm{i})}(x) = -V^{(\mathrm{i})}(-x) \in \mathbb{R}$ and configuration space $\mathbb{R}$ (or a subset of it), that have a Hermitian counterpart

$$h = H_0 + \varepsilon^2 h^{(2)} + \varepsilon^4 h^{(4)} + \cdots \tag{4}$$

which to lowest order in ε reduces to some PDM Hamiltonian, i.e.,

$$H_0 + \varepsilon^2 h^{(2)} = p\frac{1}{2m(x)}p + V_{\mathrm{eff}}(x) \tag{5}$$

with $1/m(x) = (1/m_0)[1 + \varepsilon^2 M^{(2)}(x)]$, $V_{\mathrm{eff}}(x) = V^{(\mathrm{r})}(x) + \varepsilon^2 V^{(2)}_{\mathrm{eff}}(x)$ and $M^{(2)}(x)$, $V^{(2)}_{\mathrm{eff}}(x) \in \mathbb{R}$. It should be noted that the right-hand side of (4) only contains even powers of ε because the coefficients of odd powers have been shown to vanish [3, 4], while the right-hand side of (5) is the most general expression of Hermitian PDM Hamiltonians [8]. The latter is written in terms of an effective potential $V_{\mathrm{eff}}(x)$ including some mass terms depending on two ambiguity parameters, which take the non-commutativity of the momentum and PDM operators into account [14].

It proves convenient to introduce dimensionless quantities defined by

$$\begin{aligned} &\mathrm{x} = \ell^{-1}x \qquad \mathrm{p} = \ell\hbar^{-1}p \\ &\mathrm{H} = \nu^{-1}H = \mathrm{H}_0 + \varepsilon\mathrm{H}_1 \qquad \mathrm{H}_0 = \tfrac{1}{2}\mathrm{p}^2 + \mathrm{V}^{(\mathrm{r})}(\mathrm{x}) \qquad \mathrm{H}_1(\mathrm{x}) = \mathrm{i}\mathrm{V}^{(\mathrm{i})}(\mathrm{x}) \\ &\mathrm{h} = \nu^{-1}h = \mathrm{H}_0 + \varepsilon^2\mathrm{h}^{(2)} = \tfrac{1}{2}\mathrm{p}[1 + \varepsilon^2\mathrm{M}^{(2)}(\mathrm{x})]\mathrm{p} + \mathrm{V}^{(\mathrm{r})}(\mathrm{x}) + \varepsilon^2\mathrm{V}^{(2)}_{\mathrm{eff}}(\mathrm{x}) \end{aligned} \tag{6}$$

in terms of some length and energy scales, ℓ and $\nu = \hbar^2/(m_0\ell^2)$. Note that in (3), (4) and (5), ε is also dimensionless, as well as $M^{(2)}(x)$.

In [3, 4] (see also [6]), it has been shown that for the positive-definite metric operator η_+, one may take

$$\eta_+ = \mathrm{e}^{-Q(\mathrm{x},\mathrm{p})} \qquad Q(\mathrm{x},\mathrm{p}) = \varepsilon Q_1(\mathrm{x},\mathrm{p}) + \varepsilon^3 Q_3(\mathrm{x},\mathrm{p}) + \cdots \tag{7}$$

where every $Q_j(\mathrm{x},\mathrm{p})$, $j = 1, 3, \ldots$, is such that $Q_j(\mathrm{x},\mathrm{p}) = Q_j^\dagger(\mathrm{x},\mathrm{p}) = Q_j(-\mathrm{x},\mathrm{p}) = -Q_j(\mathrm{x},-\mathrm{p})$. Then to lowest order in ε, equations (1) and (2) lead to the two conditions

$$[\mathrm{H}_0, Q_1] = -2\mathrm{H}_1 \qquad \tfrac{1}{4}[\mathrm{H}_1, Q_1] = \mathrm{h}^{(2)} \tag{8}$$

which in the case of (3) and (5) amount to

$$\left[\frac{1}{2}\mathrm{p}^2 + \mathrm{V}^{(\mathrm{r})}(\mathrm{x}), Q_1\right] = -2\mathrm{i}\mathrm{V}^{(\mathrm{i})}(\mathrm{x}) \tag{9}$$

$$\frac{\mathrm{i}}{4}[\mathrm{V}^{(\mathrm{i})}(\mathrm{x}), Q_1] = \frac{1}{2}\mathrm{p}\mathrm{M}^{(2)}(\mathrm{x})\mathrm{p} + \mathrm{V}^{(2)}_{\mathrm{eff}}(\mathrm{x}). \tag{10}$$

For Q_1, let us choose a general ansatz somewhat different from those previously considered:

$$Q_1 = \sum_{k=0}^{\infty}\{R_k(\mathrm{x}), \mathrm{p}^{2k+1}\} \qquad R_k(\mathrm{x}) = R_k(-\mathrm{x}). \tag{11}$$

By expressing p as $-\mathrm{i}\mathrm{d}/\mathrm{d}\mathrm{x}$ and using the commutation relation,

$$\left[\frac{\mathrm{d}^k}{\mathrm{d}\mathrm{x}^k}, f(\mathrm{x})\right] = \sum_{l=0}^{k-1}\binom{k}{l}\frac{\mathrm{d}^{k-l}f(\mathrm{x})}{\mathrm{d}\mathrm{x}^{k-l}}\frac{\mathrm{d}^l}{\mathrm{d}\mathrm{x}^l}, \tag{12}$$

Q_1 can be written in normal form, i.e., with all functions of x on the left of the differential operators, as

$$Q_1 = -\mathrm{i}\sum_{k=0}^{\infty} S_k(\mathrm{x})\frac{\mathrm{d}^k}{\mathrm{d}\mathrm{x}^k} \tag{13}$$

where

$$\begin{aligned} S_{2k} &= \sum_{l=k}^{\infty}(-1)^l\binom{2l+1}{2k}\frac{\mathrm{d}^{2l-2k+1}R_l}{\mathrm{d}\mathrm{x}^{2l-2k+1}} \\ S_{2k+1} &= \sum_{l=k}^{\infty}(1+\delta_{l,k})(-1)^l\binom{2l+1}{2k+1}\frac{\mathrm{d}^{2l-2k}R_l}{\mathrm{d}\mathrm{x}^{2l-2k}} \qquad \text{for} \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{14}$$

On inserting (13) in (9) and (10) and employing (12) again, we find after some straightforward calculations that equation (9) is equivalent to the conditions

$$\frac{1}{2}\frac{\mathrm{d}^2 S_0}{\mathrm{d}\mathrm{x}^2} + \sum_{l=1}^{\infty} S_l\frac{\mathrm{d}^l\mathrm{V}^{(\mathrm{r})}}{\mathrm{d}\mathrm{x}^l} = -2\mathrm{V}^{(\mathrm{i})} \tag{15}$$

$$\frac{\mathrm{d}S_{k-1}}{\mathrm{d}\mathrm{x}} + \frac{1}{2}\frac{\mathrm{d}^2 S_k}{\mathrm{d}\mathrm{x}^2} + \sum_{l=k+1}^{\infty}\binom{l}{k} S_l\frac{\mathrm{d}^{l-k}\mathrm{V}^{(\mathrm{r})}}{\mathrm{d}\mathrm{x}^{l-k}} = 0, \qquad k = 1, 2, \ldots, \tag{16}$$

while equation (10) leads to

$$\sum_{l=1}^{\infty} S_l\frac{\mathrm{d}^l\mathrm{V}^{(\mathrm{i})}}{\mathrm{d}\mathrm{x}^l} = -4\mathrm{V}^{(2)}_{\mathrm{eff}} \tag{17}$$

$$\sum_{l=2}^{\infty} \binom{l}{1} S_l \frac{\mathrm{d}^{l-1}\mathrm{V}^{(\mathrm{i})}}{\mathrm{d}\mathrm{x}^{l-1}} = 2\frac{\mathrm{d}\mathrm{M}^{(2)}}{\mathrm{d}\mathrm{x}} \tag{18}$$

$$\sum_{l=3}^{\infty} \binom{l}{2} S_l \frac{\mathrm{d}^{l-2}\mathrm{V}^{(\mathrm{i})}}{\mathrm{d}\mathrm{x}^{l-2}} = 2\mathrm{M}^{(2)} \tag{19}$$

$$\sum_{l=k+1}^{\infty} \binom{l}{k} S_l \frac{\mathrm{d}^{l-k}\mathrm{V}^{(\mathrm{i})}}{\mathrm{d}\mathrm{x}^{l-k}} = 0 \qquad k = 3, 4, \ldots. \tag{20}$$

To be able to solve the general equations (15)–(20), it is appropriate to make some simplifying assumption. Inspired by the example of the $\mathcal{PT}$-symmetric cubic anharmonic oscillator considered in [3, 4], where Q_1 only contains linear and cubic powers of p, let us assume that $R_k(\mathrm{x}) = 0$, $k = 2, 3, \ldots$, in equation (11). It then follows from (14) that only the first four functions S_k in the expansion (13) may be non-vanishing and that they are given in terms of R_0, R_1, and their derivatives by $S_0 = R_0' - R_1'''$, $S_1 = 2R_0 - 3R_1''$, $S_2 = -3R_1'$ and $S_3 = -2R_1$.

Let us first solve equations (15) and (16). In the latter, k is now restricted to $k \leqslant 4$. For $k = 4$, we obtain that S_3 must be a constant, this implying that

$$R_1(\mathrm{x}) = c_1. \tag{21}$$

Hence the remaining non-vanishing S_k are

$$S_0 = R_0' \qquad S_1 = 2R_0 \qquad S_3 = -2c_1. \tag{22}$$

From equation (16) with $k = 2$, we get

$$R_0(\mathrm{x}) = 3c_1\mathrm{V}^{(\mathrm{r})}(\mathrm{x}) + c_0 \tag{23}$$

where c_0 is another integration constant, while the equations with $k = 1$ or $k = 3$ are automatically satisfied. Equation (15) then provides us with a condition on $\mathrm{V}^{(\mathrm{i})}$,

$$\mathrm{V}^{(\mathrm{i})}(\mathrm{x}) = \tfrac{1}{4}c_1\mathrm{V}^{(\mathrm{r})'''}(\mathrm{x}) - [3c_1\mathrm{V}^{(\mathrm{r})}(\mathrm{x}) + c_0]\mathrm{V}^{(\mathrm{r})'}(\mathrm{x}). \tag{24}$$

Let us next turn to equations (17)–(20). It is easy to see that only equations (17) and (19) impose some new conditions, namely

$$\mathrm{M}^{(2)}(\mathrm{x}) = -3c_1\mathrm{V}^{(\mathrm{i})'}(\mathrm{x}) \qquad \mathrm{V}^{(2)}_{\mathrm{eff}}(\mathrm{x}) = \tfrac{1}{2}\{-[3c_1\mathrm{V}^{(\mathrm{r})}(\mathrm{x}) + c_0]\mathrm{V}^{(\mathrm{i})'}(\mathrm{x}) + c_1\mathrm{V}^{(\mathrm{i})'''}(\mathrm{x})\}, \tag{25}$$

where $\mathrm{V}^{(\mathrm{i})}(\mathrm{x})$ must be expressed in terms of $\mathrm{V}^{(\mathrm{r})}(\mathrm{x})$ through equation (24). This completes the solution of equations (9) and (10).

It is then straightforward to go back to x, p and unscaled operators. This leads to the conclusion that there exists a whole class of Hermitian PDM Hamiltonians, which to lowest order of perturbation theory allow an equivalent $\mathcal{PT}$-symmetric description and might therefore be easier to deal with than generic ones. The various members of the class are distinguished by the choice of the zeroth-order part $V^{(\mathrm{r})}(x)$ of the effective potential $V^{(2)}_{\mathrm{eff}}(x)$ and that of two integration constants c_0, c_1. The lowest-order corrections to the mass term $M^{(2)}(x)$ and to the effective potential in the PDM equation, as well as the imaginary part $V^{(\mathrm{i})}(x)$ of the corresponding $\mathcal{PT}$-symmetric potential, are indeed entirely fixed by such a choice.

The classical Hamiltonians $H_{\mathrm{c}}(x_{\mathrm{c}}, p_{\mathrm{c}})$ corresponding to the members of this class can be obtained by replacing x and p in h by the classical variables x_{c} and p_{c} and evaluating the resulting expressions in the limit $\hbar \to 0$ (assuming this limit exists), i.e., $H_{\mathrm{c}}(x_{\mathrm{c}}, p_{\mathrm{c}}) = \lim_{\hbar\to 0} h(x_{\mathrm{c}}, p_{\mathrm{c}})$.

The η_+-pseudo-Hermitian position and momentum operators X and P, as well as the physical wavefunctions $\Psi(x)$, can be calculated in the same way as h. To second order in ε, the pseudo-Hermitian operators are given by

$$O = o - \tfrac{1}{2}\varepsilon[o, Q_1] + \tfrac{1}{8}\varepsilon^2[[o, Q_1], Q_1] \qquad o = x \text{ or } p. \tag{26}$$

For the dimensionless operators, we find

$$\begin{aligned} &[\mathrm{x}, Q_1] = \mathrm{i}\sum_{k=0}^{\infty}(k+1)S_{k+1}\frac{\mathrm{d}^k}{\mathrm{dx}^k} \qquad [[\mathrm{x}, Q_1], Q_1] = \sum_{k=0}^{\infty} T_k \frac{\mathrm{d}^k}{\mathrm{dx}^k} \\ &[\mathrm{p}, Q_1] = -\sum_{k=0}^{\infty}\frac{\mathrm{d}S_k}{\mathrm{dx}}\frac{\mathrm{d}^k}{\mathrm{dx}^k} \qquad [[\mathrm{p}, Q_1], Q_1] = \mathrm{i}\sum_{k=0}^{\infty} U_k \frac{\mathrm{d}^k}{\mathrm{dx}^k} \end{aligned} \tag{27}$$

where T_k and U_k are defined by

$$T_k = \sum_{l=0}^{k}\sum_{m=k-l+1}^{\infty} T_k^{(l,m)} \qquad U_k = \sum_{l=0}^{k}\sum_{m=k-l+1}^{\infty} U_k^{(l,m)} \tag{28}$$

with

$$\begin{aligned} T_k^{(l,m)} &= \binom{m}{k-l}\left[(m+1)S_{m+1}\frac{\mathrm{d}^{l+m-k}S_l}{\mathrm{dx}^{l+m-k}} - (l+1)S_m\frac{\mathrm{d}^{l+m-k}S_{l+1}}{\mathrm{dx}^{l+m-k}}\right] \\ U_k^{(l,m)} &= \binom{m}{k-l}\left[\frac{\mathrm{d}S_m}{\mathrm{dx}}\frac{\mathrm{d}^{l+m-k}S_l}{\mathrm{dx}^{l+m-k}} - S_m\frac{\mathrm{d}^{l+m-k+1}S_l}{\mathrm{dx}^{l+m-k+1}}\right]. \end{aligned} \tag{29}$$

Similarly, the physical wavefunctions can be expressed as

$$\Psi(\mathrm{x}) = \psi(x) - \frac{\varepsilon}{2}\langle \mathrm{x}|Q_1|\psi\rangle + \frac{\varepsilon^2}{8}\langle \mathrm{x}|Q_1^2|\psi\rangle \tag{30}$$

where Q_1 is given by (13) and

$$Q_1^2 = -\sum_{k=0}^{\infty} W_k(\mathrm{x})\frac{\mathrm{d}^k}{\mathrm{dx}^k} \tag{31}$$

with

$$W_k = \sum_{l=0}^{k}\sum_{m=k-l}^{\infty} W_k^{(l,m)} \qquad W_k^{(l,m)} = \binom{m}{k-l} S_m \frac{\mathrm{d}^{l+m-k}S_l}{\mathrm{dx}^{l+m-k}}. \tag{32}$$

With the simplifying assumption (22) and taking equations (23) and (26)–(32) into account, we obtain

$$\begin{aligned} \mathrm{X} = \mathrm{x} &- \mathrm{i}\varepsilon(3c_1\mathrm{V}^{(\mathrm{r})} + c_0 + 3c_1\mathrm{p}^2) + \tfrac{3}{4}\varepsilon^2 c_1[-c_1(6\mathrm{V}^{(\mathrm{r})}\mathrm{V}^{(\mathrm{r})\prime} + \mathrm{V}^{(\mathrm{r})\prime\prime\prime}) \\ &- 2c_0\mathrm{V}^{(\mathrm{r})\prime} - 6\mathrm{i}c_1\mathrm{V}^{(\mathrm{r})\prime\prime}\mathrm{p} + 6c_1\mathrm{V}^{(\mathrm{r})\prime}\mathrm{p}^2] \end{aligned} \tag{33}$$

$$\begin{aligned} \mathrm{P} = \mathrm{p} &+ \tfrac{3}{2}\varepsilon c_1(\mathrm{V}^{(\mathrm{r})\prime\prime} + 2\mathrm{i}\mathrm{V}^{(\mathrm{r})\prime}\mathrm{p}) + \tfrac{3}{4}\mathrm{i}\varepsilon^2 c_1\{c_1(3\mathrm{V}^{(\mathrm{r})\prime}\mathrm{V}^{(\mathrm{r})\prime\prime} - 3\mathrm{V}^{(\mathrm{r})}\mathrm{V}^{(\mathrm{r})\prime\prime\prime} + \mathrm{V}^{(\mathrm{r})\prime\prime\prime\prime\prime}) \\ &- c_0 V^{(\mathrm{r})\prime\prime\prime} + \mathrm{i}[c_1(6V^{(\mathrm{r})\prime 2} - 6\mathrm{V}^{(\mathrm{r})}\mathrm{V}^{(\mathrm{r})\prime\prime} + 5\mathrm{V}^{(\mathrm{r})\prime\prime\prime\prime}) \\ &- 2c_0\mathrm{V}^{(\mathrm{r})\prime\prime}]\mathrm{p} - 9c_1\mathrm{V}^{(\mathrm{r})\prime\prime\prime}\mathrm{p}^2 - 6\mathrm{i}c_1\mathrm{V}^{(\mathrm{r})\prime\prime}\mathrm{p}^3\} \end{aligned} \tag{34}$$

and

$$\begin{aligned} \Psi(\mathrm{x}) = \psi(x) &+ \frac{1}{2}\mathrm{i}\varepsilon\left[3c_1\mathrm{V}^{(\mathrm{r})\prime} + 2(3c_1\mathrm{V}^{(\mathrm{r})} + c_0)\frac{\mathrm{d}}{\mathrm{dx}} - 2c_1\frac{\mathrm{d}^3}{\mathrm{dx}^3}\right] \\ &- \frac{\varepsilon^2}{8}\left\{3c_1[c_1(3\mathrm{V}^{(\mathrm{r})\prime 2} + 6\mathrm{V}^{(\mathrm{r})}\mathrm{V}^{(\mathrm{r})\prime\prime} - 2\mathrm{V}^{(\mathrm{r})\prime\prime\prime\prime}) + 2c_0\mathrm{V}^{(\mathrm{r})\prime\prime}]\right. \end{aligned}$$

$$+ 6c_1[c_1(12\mathrm{V}^{(\mathrm{r})}\mathrm{V}^{(r)\prime} - 5\mathrm{V}^{(\mathrm{r})\prime\prime\prime}) + 4c_0\mathrm{V}^{(r)\prime}]\frac{\mathrm{d}}{\mathrm{dx}} + 2\big[9c_1^2(2\mathrm{V}^{(\mathrm{r})2} - 3\mathrm{V}^{(\mathrm{r})\prime\prime})$$
$$+ 12c_0c_1\mathrm{V}^{(\mathrm{r})} + 2c_0^2\big]\frac{\mathrm{d}^2}{\mathrm{dx}^2} - 48c_1^2\mathrm{V}^{(\mathrm{r})\prime}\frac{\mathrm{d}^3}{\mathrm{dx}^3} - 8c_1(3c_1\mathrm{V}^{(\mathrm{r})} + c_0)\frac{\mathrm{d}^4}{\mathrm{dx}^4} + 4c_1^2\frac{\mathrm{d}^6}{\mathrm{dx}^6}\bigg\}. \tag{35}$$

It is easy to check that, as expected, the Hermitian PDM quartic anharmonic oscillator of [3, 4] belongs to the class of Hermitian PDM Hamiltonians with an approximate $\mathcal{PT}$-symmetric counterpart. On setting $\mathrm{V}^{(\mathrm{r})}(\mathrm{x}) = \frac{1}{2}\mathcal{M}^2\mathrm{x}^2, c_0 = 0$ and $c_1 = -2/(3\mathcal{M}^4)$ in equation (24), where the dimensionless quantitites are defined as in equations (17)–(20) of [4], we indeed obtain $\mathrm{V}^{(\mathrm{i})}(\mathrm{x}) = \mathrm{x}^3$, so that $V^{(\mathrm{i})}(x) = x^3$. Furthermore, from equations (25), (33) and (34), we obtain $m(x) = m_0[1 + 6(\epsilon^2/\mu^4)x^2]^{-1}$, $V_{\mathrm{eff}}^{(2)}(x) = (3m_0\mu^2x^4 - 4\hbar^2)/(2m_0\mu^4)$, $X = x + \mathrm{i}(\epsilon/\mu^4)(\mu^2x^2 + 2p^2/m_0) + (\epsilon^2/\mu^6)(-\mu^2x^3 - 2\mathrm{i}\hbar p/m_0 + 2xp^2/m_0)$ and $P = p - \mathrm{i}(\epsilon/\mu^2)(2xp - \mathrm{i}\hbar) + (\epsilon^2/\mu^6)(2p^3/m_0 - \mu^2x^2p + \mathrm{i}\hbar\mu^2x)$, which after some reordering agree with [3, 4], as does the classical Hamiltonian. Similarly, equation (35) gives rise to equation (65) of [4].

A new example is provided by selecting for $V^{(\mathrm{r})}(x)$ a one-parameter trigonometric Pöschl–Teller potential [15]

$$V^{(\mathrm{r})}(x) = V_0 \sec^2 kx \qquad V_0 = \frac{\hbar^2k^2}{2m^2}\lambda(\lambda - 1) \quad \lambda > 2 \tag{36}$$

on the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$. On setting $\ell = k^{-1}$ and $\nu = \hbar^2k^2/m_0$ for the length and energy scales, respectively, we obtain the dimensionless quantities $\mathrm{x} = kx$, $\mathrm{p} = p/(\hbar k)$ and $\mathrm{V}^{(\mathrm{r})}(\mathrm{x}) = \frac{1}{2}\lambda(\lambda - 1)\sec^2\mathrm{x}$ with $\lambda(\lambda - 1) = 2V_0/\nu$.

The choice $c_0 = -c_1 = \frac{1}{3}$ in (24) leads to

$$\mathrm{V}^{(\mathrm{i})}(\mathrm{x}) = \frac{1}{2}(\lambda + 1)\lambda(\lambda - 1)(\lambda - 2)\sec^4\mathrm{x}\tan\mathrm{x} = \frac{2}{\nu^2}V_0(V_0 - \nu)\sec^4\mathrm{x}\tan\mathrm{x}. \tag{37}$$

This means that the corresponding $\mathcal{PT}$-symmetric Hamiltonian may be written as

$$H = \frac{p^2}{2m_0} + V_0\sec^2 kx + \mathrm{i}\epsilon\sec^4 kx\tan kx \tag{38}$$

where ϵ has the dimension of an energy and is given in terms of the dimensionless ε by $\epsilon = 2\varepsilon V_0(V_0 - \nu)/\nu$.

To second order in ϵ, such a non-Hermitian Hamiltonian is equivalent to a Hermitian PDM one, given by equation (5), where

$$m(x) = m_0\left(1 + \frac{\epsilon^2}{2V_0(V_0 - \nu)}\sec^4 kx(5\sec^2 kx - 4)\right)^{-1} \tag{39}$$

and

$$V_{\mathrm{eff}}(x) = V_0\sec^2 kx + \frac{\epsilon^2}{4V_0(V_0 - \nu)}\sec^4 kx[5(V_0 - 14\nu)\sec^4 kx$$
$$- (4V_0 - 85\nu)\sec^2 kx - 20\nu]. \tag{40}$$

The corresponding η_+-pseudo-Hermitian position and momentum operators can be expressed as

$$X = x - \mathrm{i}\frac{\epsilon}{2kV_0(V_0 - \nu)}\left(-V_0\sec^2 kx + \frac{\nu}{3} - \frac{p^2}{m_0}\right) - \frac{\epsilon^2}{4kV_0(V_0 - \nu)^2}\sec^2 kx$$
$$\times\left\{[(V_0 + 2\nu)\sec^2 kx - \nu]\tan kx + \mathrm{i}\sqrt{\frac{\nu}{m_0}}(3\sec^2 kx - 2)p - \tan kx\frac{p^2}{m_0}\right\} \tag{41}$$

$$P = p - \frac{\epsilon}{2(V_0 - \nu)} \sec^2 kx[\sqrt{m_0\nu}(3\sec^2 kx - 2) + 2\mathrm{i}\tan kxp] - \mathrm{i}\frac{\epsilon^2}{4V_0(V_0-\nu)^2}\sec^2 kx \times \left\{\sqrt{m_0\nu}[3V_0\sec^4 kx - 2\nu(30\sec^4 kx - 19\sec^2 kx + 1)] + \mathrm{i}[V_0\sec^4 kx - \nu(50\sec^4 kx - 49\sec^2 kx + 6)]p + 6\sqrt{\frac{\nu}{m_0}}(3\sec^2 kx - 1)\tan kxp^2 + \frac{\mathrm{i}}{m_0}(3\sec^2 kx - 2)p^3\right\}. \tag{42}$$

Similar results can be found for physical wavefunctions. For lack of space, let us only mention the result in dimensionless variable obtained for the function $\psi(\mathrm{x}) = \cos^\lambda(\mathrm{x})$ (corresponding to the ground state of the real potential (36)):

$$\Psi(\mathrm{x}) = \cos^\lambda(\mathrm{x})\left\{1 + \frac{\mathrm{i}}{6}\varepsilon(\lambda+1)\lambda(\lambda-1)(\sec^2\mathrm{x}+2)\tan\mathrm{x} - \frac{\varepsilon^2}{72}(\lambda+1)\lambda(\lambda-1)[(\lambda-4) \times(\lambda-2)(\lambda+15)\sec^6\mathrm{x} + 3(\lambda-2)(\lambda^2-4\lambda+15)\sec^4\mathrm{x} - 4(\lambda+1)\lambda(\lambda-1)\right\}. \tag{43}$$

In the classical limit, ν goes to zero. To get a non-vanishing limit for V_0, we must therefore assume that λ goes to infinity as $\hbar^{-1}$ (this implying, in particular, that λ becomes negligeably small compared with λ^2). To second order in ϵ, the classical Hamiltonian corresponding to (38) is obtained as

$$H_\mathrm{c} = \frac{p_\mathrm{c}^2}{2m(x_\mathrm{c})} + V_0\sec^2 kx_\mathrm{c} + \frac{\epsilon^2}{4V_0}\sec^6 kx_\mathrm{c}(5\sec^2 kx_\mathrm{c} - 4) \tag{44}$$

where

$$m_\mathrm{c}(x_\mathrm{c}) = m_0\left(1 - \frac{\epsilon^2}{2V_0^2}\sec^4 kx_\mathrm{c}(5\sec^2 kx_\mathrm{c} - 4)\right) \tag{45}$$

while the classical η_+-pseudo-Hermitian variables X_c, P_c are

$$X_\mathrm{c} = x_\mathrm{c} + \mathrm{i}\frac{\epsilon}{2kV_0^2}\left(V_0\sec^2 kx_\mathrm{c} + \frac{p_\mathrm{c}^2}{m_0}\right) - \frac{\epsilon^2}{4kV_0^3}\sec^2 kx_\mathrm{c}\left(V_0\sec^2 kx_\mathrm{c} - \frac{p_\mathrm{c}^2}{m_0}\right)\tan kx_\mathrm{c} \tag{46}$$

$$P_\mathrm{c} = p_\mathrm{c} - \mathrm{i}\frac{\epsilon}{V_0}\sec^2 kx_\mathrm{c}\tan kx_\mathrm{c}p_\mathrm{c} + \frac{\epsilon^2}{4V_0^3}\sec^2 kx_\mathrm{c}\left[V_0\sec^4 kx_\mathrm{c} + (3\sec^2 kx_\mathrm{c} - 2)\frac{p_\mathrm{c}^2}{m_0}\right]p_\mathrm{c}. \tag{47}$$

It is worth noting that in contrast with what happens for the $\mathcal{PT}$-symmetric cubic anharmonic oscillator, the operators X and P involve $\hbar$ even after rewritting them in a symmetrized form. As a consequence, the η_+-pseudo-Hermitian quantization of the classical Hamiltonian (44) is far from trivial. This illustrates the importance of the factor-ordering problem in pseudo-Hermitian quantum mechanics.

In conclusion, the generalization of the works in [3, 4] that we have proposed here contributes to exploring further the relationships between $\mathcal{PT}$-symmetric and Hermitian PDM Hamiltonians started there and continued in [7, 16]. Moreover, it suggests the interest of performing detailed calculations for some new $\mathcal{PT}$-symmetric systems, such as the one defined in (38).

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